

# Unitary Positive-Energy Representations of Scalar Bilocal Quantum Fields

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## Abstract

The superselection sectors of two classes of scalar bilocal quantum fields in  $D \geq 4$  dimensions are explicitly determined by working out the constraints imposed by unitarity. The resulting classification in terms of the dual of the respective gauge groups  $U(N)$  and  $O(N)$  confirms the expectations based on general results obtained in the framework of local nets in algebraic quantum field theory, but the approach using standard Lie algebra methods rather than abstract duality theory is complementary. The result indicates that one does not lose interesting models if one postulates the absence of scalar fields of dimension  $D - 2$  in models with global conformal invariance. Another remarkable outcome is the observation that, with an appropriate choice of the Hamiltonian, a Lie algebra embedded into the associative algebra of observables completely fixes the representation theory.

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## 1 Introduction

An important tool in the study of *globally conformal invariant* (GCI) quantum field theory models in even space-time dimensions  $D \geq 4$  [21, 19, 20, 18] are bilocal fields, which arise in operator product expansions (OPE) as follows.

Let  $\phi(x)$  be a local scalar field of dimension  $d \geq 2d_0 := D - 2$ . We denote the contribution of twist  $D - 2$  in the OPE of  $((x_1 - x_2)^2)^{d-d_0} \cdot \phi^*(x_1)\phi(x_2)$  by

$$W(x_1, x_2) = W(x_2, x_1)^* \quad (1.1)$$

if  $\phi(x)$  is *complex*, and by

$$V(x_1, x_2) = V(x_2, x_1) = V(x_1, x_2)^* \quad (1.2)$$

if  $\phi(x) = \phi^*(x)$  is *real*. This means that the expansion of  $W(x_1, x_2)$  into local fields yields an infinite series of conserved symmetric traceless tensor currents, starting with the scalar field  $W(x, x)$  of dimension  $2d_0$  (which may be zero), and including the stress-energy tensor. The similar expansion of  $V(x_1, x_2)$  contains only tensors of even rank. It is a nontrivial consequence of GCI that the twist  $D - 2$  fields  $W$  and  $V$  are *bilocal* in the sense of (strong = Huygens) local commutativity with respect to both arguments [17].

Depending on the scaling dimension  $d$  of the field  $\phi(x)$  whose OPE produces the bilocal field, the latter may exhibit different singularities in its correlation functions. In the case  $d = D - 2$  when the singularities have the lowest possible degree, one can derive the commutation relations of the bilocal field, and they involve another bilocal field with the same properties. If we assume uniqueness of the bilocal field, then it can be normalized in such a way that the commutation relations take the form

$$[W(x_1, x_2), W(x_3, x_4)] = \Delta_{2,3} W(x_1, x_4) + \Delta_{1,4} W(x_3, x_2) + N \cdot \Delta_{12,34} \quad (1.3)$$

in the complex case, and

$$\begin{aligned} [V(x_1, x_2), V(x_3, x_4)] &= \Delta_{2,3} V(x_1, x_4) + \Delta_{2,4} V(x_1, x_3) + \\ &+ \Delta_{1,4} V(x_2, x_3) + \Delta_{1,3} V(x_2, x_4) + N \cdot (\Delta_{12,34} + \Delta_{12,43}) \end{aligned} \quad (1.4)$$

in the real case, where  $\Delta_{i,j}^+ = \Delta_+(x_i - x_j)$  and  $\Delta_{i,j} = \Delta_{i,j}^+ - \Delta_{j,i}^+$  are the two-point and commutator functions of a massless free scalar field (which has dimension  $d_0$ ), and  $\Delta_{12,34} = \Delta_{1,4}^+ \Delta_{2,3}^+ - \Delta_{4,1}^+ \Delta_{3,2}^+$ . The coefficients  $N$  are the normalizations of the four-point functions

$$\begin{aligned} \langle 0 | W(x_1, x_2) W(x_3, x_4) | 0 \rangle &= N \cdot \Delta_{1,4}^+ \Delta_{2,3}^+, \\ \langle 0 | V(x_1, x_2) V(x_3, x_4) | 0 \rangle &= N \cdot (\Delta_{1,4}^+ \Delta_{2,3}^+ + \Delta_{1,3}^+ \Delta_{2,4}^+). \end{aligned} \quad (1.5)$$

The above relations are such that if  $W(x_1, x_2)$  satisfies (1.3), then  $V(x_1, x_2) = W(x_1, x_2) + W(x_2, x_1)$  satisfies (1.4) with  $2N$  instead of  $N$ .

Our goal in this paper is to explore and rule out the possibility of realizations of the above commutator relations in GCI models, other than the free field realizations

$$W(x_1, x_2) = :\vec{\varphi}^*(x_1) \cdot \vec{\varphi}(x_2): = \sum_{p=1}^N :\varphi^{p*}(x_1) \varphi^p(x_2): \quad (\text{complex}) \quad (1.6)$$

where  $\varphi^p(x)$  are  $N$  mutually commuting complex massless free fields of dimension  $d_0$ , and

$$V(x_1, x_2) = :\vec{\varphi}(x_1) \cdot \vec{\varphi}(x_2): = \sum_{p=1}^N :\varphi^p(x_1) \varphi^p(x_2): \quad (\text{real}) \quad (1.7)$$

where  $\varphi^p(x)$  are  $N$  mutually commuting real massless free fields. (Free-field constructions of bilocal fields involving spinor or vector fields can also be given; their correlations exhibit higher singularities.)

Clearly, the free field realizations exist only when  $N$  is a positive integer. Indeed, it was shown in [19] for the real case that due to Hilbert space positivity in the vacuum sector the coefficient  $N$  in (1.4) must be a positive integer and the bilocal field  $V(x_1, x_2)$  is of the form (1.7) (or  $N = 0$  corresponding to the trivial case  $V = 0$ ). As a byproduct of our considerations, we establish this result of [19] both in the complex and the real case.

In particular we deduce that, if  $\phi(x)$  is a real scalar field of dimension  $D - 2$ , which is assumed to be the unique such field in the model, and  $V(x_1, x_2)$  arises in the OPE of  $\phi(x)$  with itself, then  $V(x, x)$  is a multiple of  $\phi(x)$  and thus  $\phi(x)$  is a sum of Wick squares. Dropping the uniqueness assumption, a similar statement is still true, but then a linear combination of bilocal fields of the form (1.7) with different positive coefficients may appear [17].

It is important to observe that expressions (1.6) and (1.7) are defined only on the Fock space of the free fields or on subspaces thereof. When discussing theories containing a scalar field of dimension  $D - 2$  one has to envisage the possibility of different representations of the associated bilocal field occurring in the full Hilbert space  $\mathcal{H}$ .

A pertinent general structure theorem in the framework of algebraic QFT [7] states that all superselection sectors of a local QFT  $\mathcal{A}$  are contained in the vacuum representation of a canonically associated (graded local) field extension  $\mathcal{F}$ , and they are in a one-to-one correspondence with the irreducible unitary representations of a compact gauge group  $G$  (of the first kind) of internal symmetries of  $\mathcal{F}$ , so that  $\mathcal{A} \subset \mathcal{F}$  consists of the fixed points under  $G$ . The gauge group is unitarily implemented in the vacuum Hilbert space  $\mathcal{H}$  of  $\mathcal{F}$ , and the central projections in  $U(G)''$  decompose this Hilbert space into inequivalent representations of  $\mathcal{A} \subset U(G)'$ .<sup>1</sup>

In the above free field representation, the bilocal fields are the fixed points under the gauge group  $U(N)$  or  $O(N)$  in the theory of  $N$  free complex or real massless scalar

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<sup>1</sup> $U(G) = \{U(g) : g \in G\}$  are the unitary representers of the gauge group,  $U(G)'$  is the commutant of  $U(G)$ , i.e., the algebra of bounded operators on  $\mathcal{H}$  which commute with every element of  $U(G)$ , and  $U(G)''$  is the commutant of  $U(G)'$ , which coincides with the weak closure of the linear span of  $U(G)$ .

fields, respectively. Since the latter is known to have no nontrivial superselection sectors ([24, Sect. 3.4.5–6] and [5, App. A]), one can conclude that it coincides with the above canonical field net  $\mathcal{F}$ , so that the sectors of the bilocal field are in correspondence with the representations of  $G = \mathrm{U}(N)$  or  $\mathrm{O}(N)$ . However, the general theorem of [7] evokes a great amount of abstract group duality, and its application requires the passage from fields to nets of local algebras and back. Although this passage is well understood in the case of free fields, it would be desirable to see the comparatively simple assertion emerge by more elementary methods.

We are interested in the *unitary positive-energy* representations of the infinite dimensional Lie algebras of local commutators (1.3) and (1.4). We shall pursue two alternative formulations of the problem. The first consists in defining energy positivity with respect to the conformal Hamiltonian canonically expressed in terms of the fields (Eq. (2.4) below). In this case, we do not assume in advance the free field realizations (1.6) and (1.7) in the vacuum sector, and not even that  $N \in \mathbb{N}$  in the commutation relations; these properties will be deduced instead from unitarity (cf. [19]).

In the second formulation, in order to deal with a more general class of “additively renormalized” Hamiltonians (see the next section for details), given the free field realizations in the vacuum sector we assume that the representations are generated from the vacuum by (relatively) local fields. This allows us to use the Reeh–Schlieder theorem [23], according to which every local relation among Wightman fields which holds on the vacuum vector must hold in the full Hilbert space  $\mathcal{H}$  generated by other (relatively) local fields. Apart from the commutation relations, there are polynomial relations of the form

$$\det \left( W(x_i, x_j) \right)_{i,j=1}^{N+1} - \text{contraction terms} = 0, \quad (1.8)$$

which arise from (1.6) and (1.7) by expanding the left-hand-side of the identity

$$: \left[ \det \left( \vec{\varphi}^*(x_i) \cdot \vec{\varphi}(x_j) \right)_{i,j=1}^{N+1} \right] : = 0 \quad (1.9)$$

as a polynomial in the normal products  $:\vec{\varphi}^*(x_i) \cdot \vec{\varphi}(x_j):$  in accord with Wick’s theorem (and similarly in the real case). The determinant relations (1.8) are valid in the Fock space representation, and hence, by the Reeh–Schlieder theorem, also in other superselection sectors. Therefore, the algebra of bilocal fields may be regarded as the quotient of the associative enveloping algebra of the Lie algebra (1.3) or (1.4) by the determinant relations (1.8) and possibly some additional relations<sup>2</sup>.

We shall demonstrate that in both formulations of the problem, one arrives at the same classification of unitary positive-energy representations (superselection sectors) of the Lie algebras (1.3), (1.4) of bilocal fields. We find precisely those representations that occur in the Fock space of  $N$  free scalar fields, and they are in a one-to-one correspondence with the irreducible unitary representations of  $\mathrm{U}(N)$  in the complex case or  $\mathrm{O}(N)$  in the real case.

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<sup>2</sup>Our result shows that if additional relations exist, then they are satisfied automatically in every unitary positive-energy representation satisfying the determinant relations.

A possible application to the program of classification of GCI models is the following. If bilocal fields satisfying (1.3) or (1.4) appear in the OPE of some local fields of the model, then all other (relatively) local fields, which possibly intertwine different superselection sectors, may be regarded as Wick products of a free field multiplet transforming in some representation of  $U(N)$  or  $O(N)$ , possibly tensored with some other GCI fields that decouple from the free scalar fields. The classification problem may then be focused on subtheories with the restrictive feature that they decouple from massless scalar free fields. This is the field theoretic formulation of the result in algebraic QFT [6] that every quantum field theory extension of  $\mathcal{A}$  is contained in  $\mathcal{F} \otimes \mathcal{B}$ , where  $\mathcal{F}$  is the canonical field net associated with  $\mathcal{A}$  as above, while  $\mathcal{B}$  is an arbitrary (graded) local net.

The Lie algebras (1.3) and (1.4) are isomorphic to  $\mathfrak{u}(\infty, \infty)$  and  $\mathfrak{sp}(\infty, \mathbb{R})$ , respectively. For the classification of their unitary positive-energy representations, we use methods of highest-weight representations of finite-dimensional Lie algebras. In fact, we adapt the methods of [8, 9] developed for the proof of the Kashiwara–Vergne conjecture [14] (see below) in the finite-dimensional case, and ideas of [25] generalizing to the infinite-dimensional case.

Thus, our study relates two independent earlier developments. The first is the general insight into the structure and origin of superselection sectors within Haag’s operator algebraic approach to QFT [10], culminating in the quoted results of Doplicher and Roberts [7] and subsequent work [6]. The second is the Kashiwara–Vergne conjecture [14], proved by Enright and Parthasarathy [8] and by Jakobsen [11], according to which all unitary highest-weight representations of certain reductive Lie algebras occur in tensor products of the Segal–Shale–Weil representation. This amply generalizes the seminal method of Jordan [12] and Schwinger [26] to embed the angular momentum algebra  $\mathfrak{su}(2)$  into the algebra of two real harmonic oscillators. In the present context, it is the analog (for finitely many degrees of freedom) of our free field Fock representation.

It should be noted, however, that our assumptions do not precisely match those of [14, 8, 11]. In the latter, the authors enforce half-integrality of the Cartan spectrum by demanding integrability of the representations of  $\mathfrak{u}(n, n)$  and  $\mathfrak{sp}(2n, \mathbb{R})$  to representations of  $U(n, n)$  and of the metaplectic group  $\text{Mp}(2n, \mathbb{R})$  (the two-fold covering of  $\text{Sp}(2n, \mathbb{R})$ ), respectively. While we have no direct quantum field theoretic motivation for such requirements, it turns out that in our approach the same constraints on the spectrum arise either by the choice of the canonical conformal Hamiltonian, or by the validity of the determinant relations (1.8) through the Reeh–Schlieder theorem.

Let us point out that all of our results hold also for spacetime dimension  $D = 1$ . In this case the bilocal field  $W(x_1, x_2)$  generates the vertex algebra  $W_{1+\infty}$  with a central charge  $-N$  (corresponding to  $+N$  in the terminology of the present article, Eq. (1.3), and corresponding to the central charge  $c = 2N$  as defined in [18] in terms of the stress-energy tensor). The representation theory of  $W_{1+\infty}$  was developed by Kac and Radul, and conclusions similar to ours were obtained in [13].

Despite the fact that we do not mention nor use conformal invariance in the main body of our paper, it should be stressed that the expansions of the bilocal fields  $W(x_1, x_2)$  and  $V(x_1, x_2)$  into local fields of twist  $D - 2$  include the *conformal stress-energy tensor*. This

implies that the conformal Lie algebra  $\mathfrak{so}(D, 2)$  is embedded in the (suitably completed and centrally extended) Lie algebras  $\mathfrak{u}(\infty, \infty)$  and  $\mathfrak{sp}(\infty, \mathbb{R})$ , thus generating the global conformal invariance of the bilocal fields. In order not to distract the reader's attention from the main line of argument, we have relegated the construction of the stress-energy tensor and of the conformal generators to Appendix B.

## 2 Classification: the complex case

We classify all irreducible unitary positive-energy representations (superselection sectors) of the Lie algebra (1.3) of the complex bilocal field  $W(x_1, x_2)$ . The completely analogous case of the real bilocal field  $V(x_1, x_2)$  will be sketched in the next section.

### 2.1 Statement of the results

We first identify the commutation relations (1.3) with the Lie algebra  $\mathfrak{u}(\infty, \infty)$  as follows. We choose an orthonormal basis of the one-particle space, i.e., a basis of functions  $f_i$  ( $i = 1, 2, \dots$ ) of positive energy such that

$$\Delta_+(\bar{f}_i, f_j) = \delta_{ij}, \quad \Delta_+(f_i, \bar{f}_j) = \Delta_+(f_i, f_j) = \Delta_+(\bar{f}_i, \bar{f}_j) = 0. \quad (2.1)$$

Smearing  $W(x_1, x_2)$  with these functions and their conjugates, we define the generators

$$\begin{aligned} X_{ij} &= W(\bar{f}_i, \bar{f}_j), & X_{ij}^* &= W(f_j, f_i), \\ E_{ij}^+ &= (E_{ji}^+)^* = W(f_i, \bar{f}_j) + \frac{N}{2} \delta_{ij}, \\ E_{ij}^- &= (E_{ji}^-)^* = W(\bar{f}_j, f_i) + \frac{N}{2} \delta_{ij}, & i, j &= 1, 2, \dots \end{aligned} \quad (2.2)$$

Then commutation relations (1.3) become equivalent to the ones of  $\mathfrak{u}(\infty, \infty)$  (considered as a Lie algebra over the complex numbers equipped with the real structure given by the conjugation properties in (2.2)):

$$\begin{aligned} [E_{ij}^+, E_{kl}^+] &= \delta_{jk} E_{il}^+ - \delta_{il} E_{kj}^+, & [E_{ij}^-, E_{kl}^-] &= \delta_{jk} E_{il}^- - \delta_{il} E_{kj}^-, & [E_{ij}^+, E_{kl}^-] &= 0, \\ [E_{ij}^+, X_{kl}^*] &= \delta_{jl} X_{ki}^*, & [E_{ij}^+, X_{kl}] &= -\delta_{il} X_{kj}, \\ [E_{ij}^-, X_{kl}^*] &= \delta_{jk} X_{il}^*, & [E_{ij}^-, X_{kl}] &= -\delta_{ik} X_{jl}, \\ [X_{ij}, X_{kl}^*] &= \delta_{ik} E_{lj}^+ + \delta_{jl} E_{ki}^-. \end{aligned} \quad (2.3)$$

These relations depend neither on the space-time dimension nor on the parameter  $N$  occurring in (1.3), which has been absorbed by the shift of the generators  $E_{ii}^\pm$ . The Lie algebra alone “ignores” its field theoretic origin (1.6). This observation could create the impression that the classification problem does not depend on the parameter  $N$  at all. In fact, the parameter  $N$  will reappear either through the additional determinant relations (1.8), or through the canonical choice of the Hamiltonian defining the condition

of positive energy, as discussed below. We say that a representation has *positive energy* if the Hamiltonian is well defined and diagonalizable, its spectrum is bounded from below and all of its eigenspaces are finite dimensional.

**Remark 1:** In spite of the fact that the Lie algebra commutation relations (2.3) are equivalent to the commutation relations (1.3) of the bilocal field  $W(x_1, x_2)$  via (2.2), it is not evident if they will fix, up to a unitary equivalence, the field action. It remains to fix in addition the action of the *central* modes. Such a situation arises, for instance, in the case of an abelian current in two-dimensional (chiral) conformal field theory. In the case at hand of a harmonic bilocal field, the d'Alembert equation entails that if a mode of the type of (2.2) is zero in the vacuum representation, then it will be zero also in any other representation where the d'Alembert equation is satisfied. But since we are interested only in representations which are locally intertwined with the vacuum sector, the d'Alembert equation does hold by virtue of the Reeh-Schlieder theorem. One can easily check the statement in the mode representation given in Appendix A. Hence, for harmonic bilocal Lie fields the Lie algebra commutation relations uniquely fix the whole theory.

The canonical *conformal* Hamiltonian associated with the free field expression (1.6) is

$$H_c = \sum_{i=1}^{\infty} \varepsilon_i \cdot (E_{ii}^+ + E_{ii}^- - N), \quad (2.4)$$

provided we choose the above orthonormal basis  $\{f_i\}$  to diagonalize the one-particle conformal Hamiltonian with eigenvalues  $\varepsilon_i$ . The latter are positive integers and occur with finite multiplicities depending on the space-time dimension. (An explicit diagonalization of the conformal Hamiltonian will be provided in Appendix A.)

We also introduce the charge operator

$$Q = \sum_{i=1}^{\infty} (E_{ii}^+ - E_{ii}^-), \quad (2.5)$$

which generates the center of the Lie algebra  $\mathfrak{u}(\infty, \infty)$ , and we demand that  $Q$  is well defined on the representation.

We define a *vacuum representation* as an irreducible unitary positive-energy representation of the commutation relations (1.3) of the bilocal field  $W(x_1, x_2)$ , in which  $Q$  is well-defined and  $H_c$  has the eigenvalue 0 on the ground state  $|0\rangle$  (the vacuum state). We shall show in Corollary 1 below that the vacuum representation exists and is unique, and that the condition  $H_c|0\rangle = 0$  is equivalent to the seemingly stronger requirement that the vacuum expectation value of  $W(x_1, x_2)$  vanishes (and similarly for  $V(x_1, x_2)$ ).

Now we can state our first main result.

**Theorem 1.** *Consider irreducible unitary positive-energy representations of the commutation relations (1.3) of the bilocal field  $W(x_1, x_2)$ , or equivalently of the Lie algebra  $\mathfrak{u}(\infty, \infty)$ , with fixed  $N$ . We assume that the charge operator (2.5) is well defined. The condition of “positive energy” is with respect to the conformal Hamiltonian  $H_c$  (2.4). Then:*

(i)  $N$  is a nonnegative integer, and all irreducible unitary positive-energy representations of  $\mathfrak{u}(\infty, \infty)$  are realized (with multiplicities) in the Fock space of  $N$  complex massless free scalar fields by (1.6).

(ii) The ground states of equivalent representations of  $\mathfrak{u}(\infty, \infty)$  in the Fock space form irreducible representations of the gauge group  $U(N)$ . This establishes a one-to-one correspondence between the irreducible representations of  $\mathfrak{u}(\infty, \infty)$  occurring in the Fock space and the irreducible representations of  $U(N)$ .

The case  $N = 0$  is included, meaning that there is only the trivial representation.

The important conclusion is that all superselection sectors of the bilocal field  $W(x_1, x_2)$  are realized in the Fock space of  $N$  complex massless free scalar fields, as anticipated following the result of [7], obtained in Haag's framework [10] using local nets of von Neumann algebras. Moreover, the multiplicity of a representation of  $W(x_1, x_2)$  in the Fock space equals the dimension of the corresponding representation of the gauge group  $U(N)$ . We shall prove Theorem 1 in the remainder of this section. In distinction to [7], our proof proceeds in a very concrete way based on the Wightman framework rather than the framework of local von Neumann algebras.

A remarkable consequence of Theorem 1 is that the determinant relations (1.9) are automatically satisfied in every unitary positive-energy representation of the Lie algebra  $\mathfrak{u}(\infty, \infty)$ .

For completeness, we display the Fock space representation (1.6) of the generators (2.2) of  $\mathfrak{u}(\infty, \infty)$ :

$$X_{ij} = \vec{b}_i \cdot \vec{a}_j, \quad 2E_{ij}^+ = \vec{a}_i^* \cdot \vec{a}_j + \vec{a}_j \cdot \vec{a}_i^*, \quad 2E_{ij}^- = \vec{b}_i^* \cdot \vec{b}_j + \vec{b}_j \cdot \vec{b}_i^*, \quad (2.6)$$

where  $\vec{a}_i = \vec{\varphi}(\bar{f}_i)$  and  $\vec{b}_i = \vec{\varphi}^*(\bar{f}_i)$  are the annihilation operators for the fields  $\vec{\varphi}$  and  $\vec{\varphi}^*$ , respectively, and the vector notation indicates that these fields are multiplets of size  $N$ . The creation operators are  $\vec{a}_i^* = \vec{\varphi}^*(f_i)$  and  $\vec{b}_i^* = \vec{\varphi}(f_i)$ , and together with the annihilation operators they satisfy the canonical commutation relations:

$$[a_i^p, a_j^{q*}] = \delta_{p,q} \delta_{i,j} = [b_i^p, b_j^{q*}], \quad [a_i^p, b_j^{q(*)}] = 0, \text{ etc.} \quad (2.7)$$

Next, we give an alternative formulation of the classification problem as follows. We start with the assumption that  $N$  is a positive integer and the bilocal field  $W(x_1, x_2)$  has the free field realization (1.6) in the vacuum sector, while all other superselection sectors are generated from the vacuum one by (relatively) local fields. We again require that the charge operator (2.5) be well defined, but now we allow a *general* Hamiltonian of the form

$$H = \sum_{i=1}^{\infty} \varepsilon_i \cdot (E_{ii}^+ + E_{ii}^- - g_i). \quad (2.8)$$

It is sufficient to assume that the energies  $\varepsilon_i$  form an increasing sequence of positive numbers,  $0 < \varepsilon_1 \leq \varepsilon_2 \leq \dots$ , with finite degeneracies. The real parameters  $g_i$  replacing the vacuum energy subtractions in (2.4) may be regarded as a (finite or infinite) additive



renormalization of the Hamiltonian. They will be adjusted in such a way that the sum (2.8) converges in the representations under consideration. We shall see in Subsect. 2.6 that, using the Reeh–Schlieder theorem, this proviso eventually fixes the parameters  $g_i$  up to a finite renormalization, which is of course irrelevant. We thus postulate that an operator (2.8) exists and is bounded from below with finite degeneracies.

To apply the Reeh–Schlieder theorem, we consider the operators

$$D_n := \det(X_{ij})_{i,j=1}^n, \quad (2.9)$$

which arise by smearing the multilocal fields  $\det(W(x_i, x_j))_{i,j=1}^n$  with the conjugates  $\bar{f}_i$  ( $i = 1, \dots, n$ ) of the first  $n$  basis functions in both arguments. In the vacuum representation, all determinant operators of the form (1.8) vanish, in particular  $D_{N+1}$  vanishes, while  $D_N \neq 0$ . (The contraction terms in (1.8) are absent in this case because all  $\bar{f}_i$  carry negative energy. Infinitely many other determinant relations are generated from  $D_{N+1}$  by taking commutators with the generators, but will not be needed for our argument.) Appealing to the Reeh–Schlieder theorem, we shall require that  $D_{N+1}$  also vanish in the unitary representations of interest, while  $D_N \neq 0$ .

**Theorem 2.** *Consider irreducible unitary positive-energy representations of the Lie algebra  $\mathfrak{u}(\infty, \infty)$  on which the operator  $D_{N+1}$  vanishes, while  $D_N \neq 0$  for some  $N \in \mathbb{N}$ . Assume that the charge operator (2.5) is well defined, and the condition of “positive energy” holds for the generalized Hamiltonian (2.8). Then the conformal Hamiltonian (2.4) is well defined and it differs from the generalized one by a finite additive constant.*

Therefore, the alternative assumptions of Theorem 2 lead to the same classification as in Theorem 1. The proof of Theorem 1 will be given in Subsect. 2.2–2.5, while Theorem 2 will be proven in Subsect. 2.6.

## 2.2 The ground state and the Cartan spectrum

We start with preliminary considerations which hold for a *general* Hamiltonian  $H$  of the form (2.8).

Consider an irreducible unitary positive-energy representation of the Lie algebra  $\mathfrak{u}(\infty, \infty)$  with commutation relations (2.3). The *Cartan subalgebra* of  $\mathfrak{u}(\infty, \infty)$  is spanned by the generators  $E_{ii}^\pm$ , which commute with each other and with the Hamiltonian  $H$ . Since, by assumption,  $H$  is diagonalizable with finite-dimensional eigenspaces, it follows that the Cartan generators can be simultaneously diagonalized.

By the commutation relations, the spectrum of the Cartan generators  $E_{ii}^\pm$  is integer-spaced and in particular discrete. A joint eigenvalue is a pair of sequences  $\underline{h}^+ = (h_1^+, h_2^+, \dots)$ ,  $\underline{h}^- = (h_1^-, h_2^-, \dots)$ . We denote such a pair by  $\underline{h} = (\underline{h}^+, \underline{h}^-)$ . On a state  $|\underline{h}\rangle$  with eigenvalues  $h_i^\pm$  of  $E_{ii}^\pm$ , the Hamiltonian  $H$  (2.8) has the eigenvalue  $\sum_i \varepsilon_i (h_i^+ + h_i^- - g_i)$ .

Since  $X_{ij}$  lowers the eigenvalues of  $H$  by  $\varepsilon_i + \varepsilon_j > 0$  and  $H$  is bounded from below, there must be a *ground state*  $|\underline{h}\rangle$  annihilated by all  $X_{ij}$ . If  $\varepsilon_i < \varepsilon_j$ , then  $E_{ij}^\pm$  lowers the eigenvalues of  $H$  by  $\varepsilon_i - \varepsilon_j$ , hence these elements also annihilate the ground state. If  $\varepsilon_i = \varepsilon_j$ ,

then  $E_{ij}^\pm |\underline{h}\rangle$  has the same eigenvalue as  $|\underline{h}\rangle$ , i.e., the ground state may be degenerate. Let an eigenvalue  $\varepsilon$  of the one-particle Hamiltonian be  $n$ -fold degenerate ( $n < \infty$ ). Then the ground states form a representation of the Lie subalgebra  $\mathfrak{u}(n) \oplus \mathfrak{u}(n)$  with generators  $E_{ij}^\pm$  ( $\varepsilon_i = \varepsilon_j = \varepsilon$ ). Choose for  $|\underline{h}\rangle$  a highest-weight vector of this representation, so that  $E_{ij}^\pm |\underline{h}\rangle = 0$  whenever  $i < j$  ( $\varepsilon_i = \varepsilon_j = \varepsilon$ ). Because  $E_{ij}^\pm$  and  $E_{kl}^\pm$  commute whenever  $\varepsilon_i = \varepsilon_j \neq \varepsilon_k = \varepsilon_l$ , the same can be done for all degenerate eigenvalues of the one-particle Hamiltonian simultaneously.

Thus the ground state  $|\underline{h}\rangle$  can be chosen to satisfy

$$X_{ij}|\underline{h}\rangle = 0 \quad \forall i, j, \quad E_{ij}^\pm |\underline{h}\rangle = 0 \quad \text{for } i < j, \quad \text{and} \quad E_{ii}^\pm |\underline{h}\rangle = h_i^\pm |\underline{h}\rangle. \quad (2.10)$$

Together with the commutation relations, the pair of sequences  $\underline{h} = (\underline{h}^+, \underline{h}^-)$  of Cartan eigenvalues determines the inner product and hence the representation completely. Unitarity imposes conditions on  $\underline{h}$ , some of which are elementary to obtain. Computing

$$\langle \underline{h} | X_{ij} X_{ij}^* | \underline{h} \rangle = h_j^+ + h_i^- \quad (2.11)$$

we deduce from unitarity that  $h_j^+ + h_i^-$  must be nonnegative for all  $i, j$ . Computing

$$\langle \underline{h} | E_{ij}^\pm E_{ji}^\pm | \underline{h} \rangle = h_i^\pm - h_j^\pm, \quad i < j, \quad (2.12)$$

we conclude that both sequences  $\underline{h}^\pm = (h_1^\pm, h_2^\pm, \dots)$  are weakly decreasing. Computing further recursively

$$\langle \underline{h} | (E_{ij}^\pm)^n (E_{ji}^\pm)^n | \underline{h} \rangle = n! (h_i^\pm - h_j^\pm)(h_i^\pm - h_j^\pm - 1) \cdots (h_i^\pm - h_j^\pm - n + 1), \quad (2.13)$$

we obtain from unitarity that all differences  $h_i^\pm - h_j^\pm$  are nonnegative integers and

$$(E_{ji}^\pm)^{h_i^\pm - h_j^\pm + 1} |\underline{h}\rangle = 0, \quad i < j. \quad (2.14)$$

Thus, the eigenvalues  $\underline{h}^\pm$  form a pair of integer-spaced weakly decreasing sequences such that  $h_i^+ + h_i^- \geq 0$ . Therefore, both sequences  $\underline{h}^+$  and  $\underline{h}^-$  must stabilize at some values  $h_\infty^+, h_\infty^-$ . The convergence of the eigenvalue  $\sum_i (h_i^+ - h_i^-)$  of  $Q$  on the ground state implies that

$$h_\infty^+ = h_\infty^- =: h_\infty \geq 0. \quad (2.15)$$

If we now specialize to the *canonical* Hamiltonian (2.4), then the convergence of the eigenvalue  $\sum_i \varepsilon_i \cdot (h_i^+ + h_i^- - N)$  of  $H_c$  implies

$$2h_\infty = N. \quad (2.16)$$

To prove that  $N$  is a nonnegative integer, let  $r$  be sufficiently large such that  $h_i^+ = h_\infty^+$  for  $i > r$ , and observe that by Eqs. (2.10), (2.12) one has  $E_{ij}^\pm |\underline{h}\rangle = h_\infty^\pm \delta_{ij} |\underline{h}\rangle$  for all  $i, j > r$ . This allows one to compute recursively the norm square of  $D_n^{(r)*} |\underline{h}\rangle$ , where

$D_n^{(r)} := \det(X_{ij})_{i,j=r+1}^{r+n}$ . Instead of computing it explicitly, it suffices to observe as in [19] that this norm square must be a polynomial  $p_n(N)$  of degree  $n$  in  $N = 2h_\infty$ , and that  $p_n(N)$  vanishes whenever  $N$  is a nonnegative integer smaller than  $n$ . Indeed, if  $N \in \mathbb{N}_0$ , the representation under consideration, restricted to the subalgebra with generators  $X_{ij}$ ,  $X_{ij}^*$  and  $E_{ij}^\pm$  with  $i, j > r$ , coincides with the vacuum representation of the free-field realization (1.6) with  $N$  complex scalar fields, where  $D_n^{(r)}$  vanishes manifestly for  $n > N$ . For the same reason, for  $n = N$ ,  $p_n(n)$  is positive. These facts taken together imply that

$$p_n(N) \sim N(N-1) \cdots (N-n+1) \quad (2.17)$$

with an irrelevant positive coefficient. Then it is clear that nonnegativity of all  $p_n(N)$  for a given value of  $N = 2h_\infty$  implies that  $N$  must be a nonnegative integer.

In particular, if we define the vacuum state requiring  $H_c|0\rangle = 0$ , we must have  $h_i^\pm = h_\infty = N/2$  for all  $i$ . Because the sequence of Cartan weights determines the representation, the vacuum representation is unique, and given by Eq. (2.6). Moreover, we have the following result.

**Corollary 1.** *The vacuum state  $|0\rangle$  spans a one-dimensional representation of the Lie subalgebra  $\mathfrak{u}(\infty) \oplus \mathfrak{u}(\infty)$  with generators  $E_{ij}^\pm$ , such that*

$$E_{ij}^\pm |0\rangle = \frac{N}{2} \delta_{ij} |0\rangle. \quad (2.18)$$

*In particular,  $Q = 0$ , and the vacuum expectation value of the bilocal field vanishes,*

$$\langle 0|W(x_1, x_2)|0\rangle = 0. \quad (2.19)$$

Note that (2.19) for a ground state  $|0\rangle$  implies  $\langle 0|H_c|0\rangle = 0$  and hence  $H_c|0\rangle = 0$  (see Proposition 2 in Appendix B). One could have expected (2.19) to be a part of the definition of the bilocal field as described in the introduction, but we see here that it follows from the seemingly weaker assumption that the vacuum state has zero energy.

### 2.3 Unitarity bounds from Casimir operators

To obtain further constraints on the admissible values of  $\underline{h}$ , we shall consider certain finite-dimensional subalgebras of  $\mathfrak{u}(\infty, \infty)$ . Namely, for a positive integer  $n$ , we consider the Lie subalgebra  $\mathfrak{u}(n, n)$  spanned by the generators (2.2) with indices  $1 \leq i, j \leq n$ . We choose  $n$  sufficiently large so that  $h_n^+ = h_n^- = h_\infty$ .

Clearly, unitarity of a representation of  $\mathfrak{u}(\infty, \infty)$  implies unitarity of its restriction to  $\mathfrak{g} := \mathfrak{u}(n, n)$ . We may then follow the strategy of [8], using results of [9]. Denote by  $\mathfrak{k} := \mathfrak{u}(n) \oplus \mathfrak{u}(n)$  the Lie algebra of the maximal compact subgroup  $U(n) \times U(n)$  of  $U(n, n)$ .

We adapt the conventions of [8] for the positive roots of these Lie algebras so that our *lowest energy* condition  $h_1^\pm \geq \cdots \geq h_n^\pm$  turns into a *highest weight* condition. Introduce an orthonormal basis  $\underline{e}_i^\pm$  ( $i = 1, \dots, n$ ) of  $\mathbb{R}^{2n}$  such that  $\underline{h} = \sum_i (h_i^+ \underline{e}_i^+ + h_i^- \underline{e}_i^-)$ . We

define the positive roots to be the roots  $\underline{e}_i^\pm - \underline{e}_j^\pm$  ( $i < j$ ) and  $-\underline{e}_i^- - \underline{e}_j^+$  associated with the annihilation operators  $E_{ij}^\pm$  ( $i < j$ ) and  $X_{ij}$  for the ground state; then the ground state  $|\underline{h}\rangle$  is a highest-weight vector for  $\mathfrak{g}$ .

Unitarity of an irreducible representation  $U_{\mathfrak{g}}(\underline{h})$  of  $\mathfrak{g}$  with a ground state  $|\underline{h}\rangle$  is equivalent to the condition that the inner product on the Verma  $\mathfrak{g}$ -module  $V_{\mathfrak{g}}(\underline{h})$  is semi-definite. Then  $U_{\mathfrak{g}}(\underline{h}) = V_{\mathfrak{g}}(\underline{h})/N_{\mathfrak{g}}(\underline{h})$  is the quotient of the Verma module by its (maximal) submodule of null vectors.

We also introduce the Verma  $\mathfrak{k}$ -module  $V_{\mathfrak{k}}(\underline{h})$  and its quotient  $U_{\mathfrak{k}}(\underline{h}) = V_{\mathfrak{k}}(\underline{h})/N_{\mathfrak{k}}(\underline{h})$  by the maximal submodule of null vectors  $N_{\mathfrak{k}}(\underline{h}) = N_{\mathfrak{g}}(\underline{h}) \cap V_{\mathfrak{k}}(\underline{h})$ . In fact,  $N_{\mathfrak{k}}(\underline{h})$  is generated by the null vectors from Eq. (2.14), and  $U_{\mathfrak{k}}(\underline{h})$  is the unitary representation of  $\mathfrak{k}$  specified as follows. Denote by  $\underline{h}^\pm$  the finite sequence  $(h_1^\pm, \dots, h_n^\pm)$ . Then we have

$$h_i^\pm - h_\infty = m_i^\pm \quad (2.20)$$

where

$$m_1^\pm \geq m_2^\pm \geq \dots \geq m_{r^\pm}^\pm > m_{r^\pm+1}^\pm = \dots = m_n = 0 \quad \text{are nonnegative integers.} \quad (2.21)$$

The restriction of the representation  $U_{\mathfrak{k}}(\underline{h})$  to  $\mathfrak{su}(n) \oplus \mathfrak{su}(n) \subset \mathfrak{k}$  is given by the pair of Young diagrams  $Y^\pm$  with  $r^\pm$  rows of length  $m_i^+$  and  $m_i^-$  ( $1 \leq i \leq r^\pm < n$ ), respectively, while  $h_\infty$  determines the representation of the center  $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$  of  $\mathfrak{k}$ .

To get a necessary condition for the unitarity of  $U_{\mathfrak{g}}(\underline{h})$ , we exploit the eigenvalues of *Casimir operators*. The Casimir operator for  $\mathfrak{k}$ ,

$$C_{\mathfrak{k}} = \sum_{ij} (E_{ij}^+ E_{ji}^+ + E_{ij}^- E_{ji}^-), \quad (2.22)$$

has an eigenvalue  $(\underline{\lambda} + \underline{\varrho}, \underline{\lambda} + \underline{\varrho}) - (\underline{\varrho}, \underline{\varrho})$  in any highest-weight representation of  $\mathfrak{k}$  with highest weight  $\underline{\lambda}$ , where  $\underline{\varrho} = \frac{1}{2} \sum_{i=1}^n (n+1-2i)(\underline{e}_i^+ + \underline{e}_i^-)$  is one-half the sum of all positive roots of  $\mathfrak{k}$ .  $(\cdot, \cdot)$  is the natural inner product in  $\mathbb{R}^{2n}$ .

On the other hand, the Casimir operator for  $\mathfrak{g}$ ,

$$C_{\mathfrak{g}} = \sum_{ij} (E_{ij}^+ E_{ji}^+ + E_{ij}^- E_{ji}^- - X_{ij}^* X_{ij} - X_{ij} X_{ij}^*), \quad (2.23)$$

has an eigenvalue  $(\underline{h} + \underline{\delta}, \underline{h} + \underline{\delta}) - (\underline{\lambda}, \underline{\lambda})$  in any highest-weight representations of  $\mathfrak{g}$  with highest weight  $\underline{h}$ , where  $\underline{\delta} = \underline{\varrho} - \frac{n}{2} \sum_{i=1}^n (\underline{e}_i^+ + \underline{e}_i^-)$ .

Then writing the difference as  $C_{\mathfrak{k}} - C_{\mathfrak{g}} = 2 \sum_{ij} X_{ij}^* X_{ij} + \sum_i (E_{ii}^+ + E_{ii}^-)$ , it is easy to calculate

$$2 \sum_{ij} \langle \underline{\lambda} | X_{ij}^* X_{ij} | \underline{\lambda} \rangle = [(\underline{\lambda} + \underline{\delta}, \underline{\lambda} + \underline{\delta}) - (\underline{h} + \underline{\delta}, \underline{h} + \underline{\delta})] \cdot \langle \underline{\lambda} | \underline{\lambda} \rangle, \quad (2.24)$$

whenever  $|\underline{\lambda}\rangle$  is a highest-weight vector for  $\mathfrak{k}$  of weight  $\underline{\lambda}$  within a highest-weight  $\mathfrak{g}$ -module with highest weight  $\underline{h}$ .

Assume now that the highest weight  $\underline{h}$  gives rise to an irreducible unitary representation of  $\mathfrak{g}$ . Then  $U_{\mathfrak{k}}(\underline{h})$  induces the highest-weight  $\mathfrak{g}$ -module  $M(\underline{h}) = \mathbb{C}[X_{kl}^*] \otimes U_{\mathfrak{k}}(\underline{h})$  spanned by

vectors of the form  $X^* \dots X^* E^\pm \dots E^\pm |\underline{h}\rangle$ , which is the quotient of the Verma module  $V_{\mathfrak{g}}(\underline{h})$  by the (nonmaximal in general) submodule  $\mathbb{C}[X_{kl}] \otimes N_{\mathfrak{k}}(\underline{h})$ .

Consider the subspace  $\mathbb{C}\{X_{kl}^*\} \otimes U_{\mathfrak{k}}(\underline{h})$  of  $M(\underline{h})$  spanned by vectors of the form  $X^* E^\pm \dots E^\pm |\underline{h}\rangle$ . This is a  $\mathfrak{k}$ -module equivalent to  $U_{\mathfrak{k}}(\underline{e}_1^+ + \underline{e}_1^-) \otimes U_{\mathfrak{k}}(\underline{h})$ , because the generators  $X_{kl}^*$  transform like a vector with respect to both  $\mathfrak{su}(n)$  factors of  $\mathfrak{k}$ . Let  $\underline{\lambda}$  be the highest weight of any  $\mathfrak{k}$ -subrepresentation of  $U_{\mathfrak{k}}(\underline{e}_1^+ + \underline{e}_1^-) \otimes U_{\mathfrak{k}}(\underline{h})$ , and let  $|\underline{\lambda}\rangle$  be the corresponding vector in  $\mathbb{C}\{X_{kl}^*\} \otimes U_{\mathfrak{k}}(\underline{h}) \subset M(\underline{h})$ .

By the unitarity assumption, expression (2.24) must be nonnegative, and if it is positive, then  $\langle \underline{\lambda} | \underline{\lambda} \rangle > 0$  and  $(\underline{\lambda} + \underline{\delta}, \underline{\lambda} + \underline{\delta}) > (\underline{h} + \underline{\delta}, \underline{h} + \underline{\delta})$ . If instead (2.24) vanishes, then  $\langle \underline{\lambda} | X_{ij}^* X_{ij} | \underline{\lambda} \rangle$  must vanish for all  $i, j$ . Since in  $M(\underline{h})$  we have  $X_{ij} U_{\mathfrak{k}}(\underline{h}) = \{0\}$ , the commutation relations (2.3) imply that the generators  $X_{ij}$  map any element of  $\mathbb{C}\{X_{kl}^*\} \otimes U_{\mathfrak{k}}(\underline{h})$  into  $U_{\mathfrak{k}}(\underline{h})$ . In particular,  $X_{ij} |\underline{\lambda}\rangle$  belongs to the Hilbert space  $U_{\mathfrak{k}}(\underline{h})$ . Hence  $X_{ij} |\underline{\lambda}\rangle = 0$ , and therefore  $|\underline{\lambda}\rangle$  is a highest-weight vector for  $\mathfrak{g}$  with weight  $\underline{\lambda}$  within  $M(\underline{h})$ . This implies that  $\langle \underline{\lambda} | \underline{\lambda} \rangle = 0$  and that  $\underline{\lambda} + \underline{\delta}$  is a  $\mathfrak{g}$ -Weyl transform of  $\underline{h} + \underline{\delta}$  [28, 3]. Then  $(\underline{\lambda} + \underline{\delta}, \underline{\lambda} + \underline{\delta}) = (\underline{h} + \underline{\delta}, \underline{h} + \underline{\delta})$ . We have obtained in both cases

$$(\underline{\lambda} + \underline{\delta}, \underline{\lambda} + \underline{\delta}) - (\underline{h} + \underline{\delta}, \underline{h} + \underline{\delta}) =: \gamma \geq 0. \quad (2.25)$$

By the Littlewood–Richardson rule,  $U_{\mathfrak{k}}(\underline{e}_1^+ + \underline{e}_1^-) \otimes U_{\mathfrak{k}}(\underline{h})$  contains  $U_{\mathfrak{k}}(\underline{\lambda})$  with  $\underline{\lambda}^\pm = \underline{h}^\pm + \underline{e}_{r^\pm+1}$  where  $r^\pm$  are the heights of the first columns of the Young diagrams  $Y^\pm$  defined by  $\underline{h}^\pm$  according to (2.21). For this choice of  $\underline{\lambda}$  we have  $\gamma = 2(2h_\infty - r^+ - r^-)$  (it can be shown that this is the minimal value of  $\gamma$ ). Then (2.25) gives the following necessary condition for unitarity:

$$r^+ + r^- \leq 2h_\infty. \quad (2.26)$$

## 2.4 Fock space representations

Kashiwara and Vergne [14] have shown that all highest-weight representations of  $\mathfrak{su}(n, n)$  satisfying the bound (2.26) with  $h_n^+ = h_n^- = h_\infty$  half-integer are contained in the  $(r^+ + r^-)$ -fold tensor power of the Segal–Shale–Weil representation, and Schmidt [25] has extended this result to  $n = \infty$ . We essentially reformulate these results in our setting.

Recall that the bilocal field  $W(x_1, x_2)$  is given on the Fock space by (1.6). Consequently, the generators (2.2) of  $\mathfrak{u}(\infty, \infty)$  are given by (2.6). Clearly, the representation of  $\mathfrak{u}(\infty, \infty)$  on the Fock space is unitary.

We claim that every representation with a ground state  $|\underline{h}\rangle$  satisfying the bound (2.26) with  $2h_\infty = N \in \mathbb{N}$  is contained in the Fock space of  $N$  complex free scalar fields. (The case  $N = 0$  implies  $r^+ = r^- = 0$ , and hence triviality of the representation by virtue of Eqs. (2.11) and (2.12).)

It is sufficient to display a vector with the properties of  $|\underline{h}\rangle$  within the Fock space. Let  $h_i^\pm = m_i^\pm + N/2$  according to (2.20) (with  $n$  sufficiently large), and let  $Y^\pm$  be the associated Young diagrams with rows of length  $m_i^\pm$ . Denote the heights of the columns of these diagrams by  $r^\pm = r_1^\pm \geq r_2^\pm \geq \dots \geq r_{m_1^\pm}^\pm$ .

Consider the Fock space vector

$$|\underline{h}\rangle_F = \left( \prod_{k=1}^{m_1^+} a^{*\wedge r_k^+} \right) \left( \prod_{l=1}^{m_1^-} b^{*\wedge r_l^-} \right) |0\rangle \quad (2.27)$$

where  $a^{*\wedge r}$  and  $b^{*\wedge r}$  stand for the components

$$a^{*\wedge r} = \det \left( a_i^{p*} \right)_{\substack{p=1,\dots,r \\ i=1,\dots,r}} \quad \text{and} \quad b^{*\wedge r} = \det \left( b_i^{p*} \right)_{\substack{p=N+1-r,\dots,N \\ i=1,\dots,r}} \quad (2.28)$$

of the antisymmetric  $U(N)$  tensors  $\vec{c}_1^* \wedge \dots \wedge \vec{c}_r^*$  ( $c = a$  or  $b$ ). Note that  $\vec{a}_i^*$  and  $\vec{b}_i^*$  transform like a  $U(N)$  vector and a conjugate vector, respectively (see Subsect. 2.5). The vector (2.27) is annihilated by  $X_{ij}$  for all  $i, j$  (because  $r_k^+ \leq N - r_l^-$  thanks to the bound (2.26) and  $2h_\infty = N$ ) and by  $E_{ij}^\pm$  for  $i < j$  (by virtue of the antisymmetrizations). In addition,  $|\underline{h}\rangle_F$  has eigenvalues  $h_i^\pm$  for the operators  $E_{ii}^\pm$ . Therefore,  $|\underline{h}\rangle_F$  has all the properties of the ground state  $|\underline{h}\rangle$ . This proves part (i) of Theorem 1.

Because the Fock space representation is unitary, the presence of this ground state in the Fock space proves, in particular, that the necessary conditions for unitarity found above are in fact also sufficient [14].

We conclude that all superselection sectors of the complex bilocal field  $W(x_1, x_2)$  are realized in the Fock space of  $N$  complex massless scalar fields. The sectors are classified by the Cartan eigenvalues of  $\mathfrak{u}(\infty, \infty)$ :

$$\begin{aligned} \underline{h}^+ &= (m_1^+ + h_\infty, \dots, m_{r^+}^+ + h_\infty, h_\infty, \dots), \\ \underline{h}^- &= (m_1^- + h_\infty, \dots, m_{r^-}^- + h_\infty, h_\infty, \dots), \end{aligned} \quad (2.29)$$

where  $h_\infty = N/2$ ,  $m_1^\pm \geq \dots \geq m_{r^\pm}^\pm > 0$  are integers, and  $r^+ + r^- \leq N$ .

## 2.5 Representations of the gauge group

It remains to relate the superselection sectors of the bilocal field  $W(x_1, x_2)$  classified in the previous subsections to the unitary representations of the gauge group  $U(N)$ .

The gauge group  $U(N)$  is unitarily represented on the Fock space in such a way that the vacuum is invariant and the creation operators  $\vec{a}^*$  and  $\vec{b}^*$  transform like an  $N$ -vector and a conjugate  $N$ -vector, respectively. In particular, the expressions (2.6) and the bilocal fields  $W(x_1, x_2)$  given by (1.6) on the Fock space are gauge invariant. Because the gauge group and the fields commute with each other, it follows that a ground state  $|\underline{h}\rangle_F$  is a component of a  $U(N)$  tensor representation whose dimension equals the multiplicity of the corresponding superselection sector within the Fock space.

The ground state  $|\underline{h}\rangle_F$  displayed in (2.27) is in fact a common highest weight vector for the commuting actions of  $\mathfrak{u}(\infty, \infty)$  and  $\mathfrak{u}(N)$  on the Fock space. The latter is the Lie algebra of the gauge group with generators

$$2E^{pq} = \sum_{i=1}^{\infty} (a_i^{p*} a_i^q + a_i^q a_i^{p*} - b_i^{q*} b_i^p - b_i^p b_i^{q*}) \quad (2.30)$$

which annihilate  $|\underline{h}\rangle_F$  if  $p < q$ . (The infinite sum in (2.30) converges on all Fock vectors on which  $Q$  is finite.)

The components  $\prod_k a^{*\wedge r_k^+}$  and  $\prod_l b^{*\wedge r_l^-}$  in (2.27) belong to tensors transforming under the gauge subgroup  $SU(N)$  in the representations given by the Young diagrams  $Y^+$  and  $(Y^-)^*$ , respectively. (The latter is the conjugate diagram whose columns have heights  $N - r_{m_1}^- \geq \dots \geq N - r_1^-$ .) The ground state  $|\underline{h}\rangle_F$  therefore belongs to a subrepresentation of the tensor product  $Y^+ \otimes (Y^-)^*$ . The tracelessness of (2.27) implies that the only contribution comes from the irreducible representation whose Young diagram  $Y$  has column heights  $N - r_{m_1}^- \geq \dots \geq N - r_1^- \geq r_1^+ \geq \dots \geq r_{m_1}^+$ , obtained as the juxtaposition of  $(Y^-)^*$  and  $Y^+$  (recall that  $N - r_1^- \geq r_1^+$  by (2.26) and  $2h_\infty = N$ ).

Similarly, because  $\vec{a}^*$  carries charge 1 and  $\vec{b}^*$  carries charge  $-1$ , the  $U(1)$  transformation is specified by the eigenvalue of the charge operator  $Q$  on  $|\underline{h}\rangle_F$  given by  $q = |Y^+| - |Y^-| = |Y| - Nm_1^-$ , where  $|Y|$  stands for the number of boxes of the diagram  $Y$  so that  $|(Y^-)^*| = Nm_1^- - |Y^-|$ . Therefore, the  $U(N)$  transformation given by the pair  $(Y, q)$  is determined by the pair  $(Y^+, Y^-)$ .

Conversely, every irreducible unitary representation of  $U(N)$  is given by a pair  $(Y, q)$  where  $|q| \leq |Y|$  and  $q = |Y| \pmod N$ . (Each  $U(N)$  vector contributes 1 to  $|Y|$  and 1 to  $q$ , while each conjugate vector contributes  $N - 1$  to  $|Y|$  and  $-1$  to  $q$ .) The pair  $(Y, q)$  then determines a unique split of  $Y$  into  $Y^+$  and  $(Y^-)^*$  such that  $q = |Y^+| - |Y^-|$ .

This gives an explicit one-to-one correspondence between the data (2.29) defining the superselection sectors and the unitary irreducible representations of the gauge group  $U(N)$ . Moreover, since the Cartan eigenvalues determine the occupation numbers in the ground state, one can see that the above ground states (2.27) exhaust the multiplicity of the superselection sector representation in the Fock space, and hence the multiplicity space carries the corresponding representation of  $U(N)$ . This proves part (ii) of Theorem 1.

## 2.6 The determinant relations

To prove Theorem 2, we follow the line of argument of Subsect. 2.2 until (2.15) for which we did not need the special choice  $H_c$  of the Hamiltonian.

We now assume that for some  $N \in \mathbb{N}$  the operator  $D_{N+1}$  vanishes in the representation under consideration, while  $D_n$  ( $n \leq N$ ) do not vanish. It is easy to compute, using expansion formulas for determinants and the commutation relations (2.3), that

$$X_{nn}D_n^*|\underline{h}\rangle = (h_n^+ + h_n^- - n + 1)D_{n-1}^*|\underline{h}\rangle. \quad (2.31)$$

Hence

$$X_{11} \cdots X_{nn}D_n^*|\underline{h}\rangle = \left( \prod_{m=1}^n (h_m^+ + h_m^- - m + 1) \right) |\underline{h}\rangle, \quad (2.32)$$

and  $D_{N+1}$  can vanish only if

$$h_m^+ + h_m^- = m - 1 \quad \text{for some positive integer } m \leq N + 1. \quad (2.33)$$

In particular, since the Cartan eigenvalues are integer-spaced and  $h_\infty^+ = h_\infty^-$ , we conclude that all Cartan eigenvalues belong to  $\frac{1}{2}\mathbb{N}_0$ . Setting  $N' := 2h_\infty$ , we obtain  $N' \leq h_m^+ + h_m^- = m - 1 \leq N$ .

Since the representation under consideration is determined by its Cartan eigenvalues  $\underline{h}^\pm$ , and  $h_\infty = N'/2$ , we know from Subsect. 2.3 and 2.4 that it is realized on the Fock space of  $N'$  complex scalar fields. But in the Fock representation,  $D_n$  vanishes if  $n > N'$  while  $D_n$  ( $n \leq N'$ ) are nontrivial. Therefore our assumption that  $D_N$  does not vanish implies that  $N \leq N'$ . We conclude that  $N' = N$  and

$$2h_\infty = N. \quad (2.34)$$

Turning to the Hamiltonian (2.8), we observe that its eigenvalue on the ground state can converge only if  $\sum_i \varepsilon_i(2h_\infty - g_i)$  is finite. Hence we must have

$$g_i = 2h_\infty + \delta_i = N + \delta_i \quad (2.35)$$

with arbitrary shifts  $\delta_i$  such that  $\Delta E = \sum_i \varepsilon_i \delta_i$  is well-defined. Clearly, this constitutes just an irrelevant additive renormalization  $H = H_c - \Delta E$  of the Hamiltonian.

This proves Theorem 2.

### 3 Classification: the real case

The classification of the superselection sectors of the real bilocal field  $V(x_1, x_2)$  satisfying (1.4) proceeds in perfect analogy to the complex case discussed in the previous section. We shall just repeat the relevant steps and point out the differences.

The generators of the Lie algebra are

$$X_{ij} = V(\bar{f}_i, \bar{f}_j) = X_{ji}, \quad X_{ij}^* \quad \text{and} \quad E_{ij} = E_{ji}^* = V(f_i, \bar{f}_j) + \frac{N}{2} \delta_{ij}. \quad (3.1)$$

They satisfy the commutator relations of  $\mathfrak{sp}(\infty, \mathbb{R})$ :

$$\begin{aligned} [E_{ij}, E_{kl}] &= \delta_{jk} E_{il} - \delta_{il} E_{kj}, \\ [E_{ij}, X_{kl}^*] &= \delta_{jk} X_{il}^* + \delta_{jl} X_{ki}^*, \quad [E_{ij}, X_{kl}] = -\delta_{ik} X_{jl} - \delta_{il} X_{kj}, \\ [X_{ij}, X_{kl}^*] &= \delta_{jk} E_{li} + \delta_{jl} E_{ki} + \delta_{ik} E_{lj} + \delta_{il} E_{kj}. \end{aligned} \quad (3.2)$$

The Fock space representation is given by

$$X_{ij} = \vec{a}_i \cdot \vec{a}_j \quad \text{and} \quad 2E_{ij} = \vec{a}_i^* \cdot \vec{a}_j + \vec{a}_j \cdot \vec{a}_i^*, \quad (3.3)$$

where  $\vec{a}_i = \vec{\varphi}(\bar{f}_i)$  and  $\vec{a}_i^* = \vec{\varphi}(f_i)$  (cf. (2.7)).

The *general* Hamiltonian is

$$H = \sum_{i=1}^{\infty} \varepsilon_i \cdot (E_{ii} - g_i), \quad (3.4)$$



while the canonical *conformal* Hamiltonian is

$$H_c = \sum_{i=1}^{\infty} \varepsilon_i \cdot \left( E_{ii} - \frac{N}{2} \right). \quad (3.5)$$

There is no charge operator in the real case. The determinant operators  $D_n$  are defined by the same formula (2.9) as in the complex case.

**Theorem 3.** *The statements of Theorems 1 and 2 hold if we replace everywhere the bilocal field  $W(x_1, x_2)$  by  $V(x_1, x_2)$ , the Lie algebra  $\mathfrak{u}(\infty, \infty)$  by  $\mathfrak{sp}(\infty, \mathbb{R})$ , complex free fields by real free fields, the gauge group  $U(N)$  by  $O(N)$ , and omit the assumption about the charge operator.*

The important conclusion is that all superselection sectors are realized in the Fock space of  $N$  real massless free scalar fields by (1.7). In the remainder of this section we give a sketch of the proof of the theorem.

The ground state  $|\underline{h}\rangle$  is annihilated by all  $X_{ij}$  and by  $E_{ij}$  for  $i < j$ . Computing the same norms as in Subsect. 2.2, we conclude that the Cartan eigenvalues of the generators  $E_{ii}$  are given by a single integer-spaced sequence  $\underline{h}$  such that  $h_1 \geq h_2 \geq \dots \geq 0$ . This sequence must stabilize at some value  $h_\infty$ . Finiteness of the canonical Hamiltonian  $H_c$  requires  $2h_\infty = N$ . Exploiting the vanishing of  $E_{ij}$  on the ground state whenever  $h_i = h_j = h_\infty$  and  $i \neq j$ , we can determine the norms of the vectors  $\det(X_{ij})_{i,j=r+1}^{r+n} |\underline{h}\rangle$  and conclude that if they are nonnegative, then  $N$  must be a nonnegative integer. The unique vacuum representation is given by (3.3), and the obvious analog of Corollary 1 holds.

For a positive integer  $n$  such that  $h_n = h_\infty$ , consider the restriction of our representation of  $\mathfrak{sp}(\infty, \mathbb{R})$  to a unitary representation of the maximal compact subalgebra  $\mathfrak{k} := \mathfrak{u}(n)$  of  $\mathfrak{g} := \mathfrak{sp}(2n, \mathbb{R}) \subset \mathfrak{sp}(\infty, \mathbb{R})$ . Then the Cartan eigenvalues have the form  $h_i = m_i + h_\infty$  ( $i \leq n$ ), where  $m_1 \geq \dots \geq m_r > 0$  are integers and  $m_{r+1} = \dots = m_n = 0$ . The Young diagram  $Y$  with rows of lengths  $m_i$  determines the representation of  $\mathfrak{su}(n) \subset \mathfrak{u}(n)$  with highest weight  $\sum m_i \underline{e}_i$ , while  $h_\infty$  determines the action of the center of  $\mathfrak{u}(n)$ . Considering the Casimir operators

$$C_{\mathfrak{k}} = \sum_{ij} E_{ij} E_{ji} \quad (3.6)$$

of  $\mathfrak{k}$  and

$$C = \sum_{ij} \left( E_{ij} E_{ji} - \frac{1}{2} (X_{ij}^* X_{ij} + X_{ij} X_{ij}^*) \right) \quad (3.7)$$

of  $\mathfrak{g}$ , one arrives at

$$\sum_{ij} \langle \underline{\lambda} | X_{ij}^* X_{ij} | \underline{\lambda} \rangle = \gamma \cdot \langle \underline{\lambda} | \underline{\lambda} \rangle, \quad \gamma = (\underline{\lambda} + \underline{\delta}, \underline{\lambda} + \underline{\delta}) - (\underline{h} + \underline{\delta}, \underline{h} + \underline{\delta}), \quad (3.8)$$

whenever  $|\underline{\lambda}\rangle$  is a highest-weight vector for  $\mathfrak{k}$  of weight  $\underline{\lambda}$  within a highest-weight  $\mathfrak{g}$ -module with highest weight  $\underline{h}$ . Here,  $\underline{\delta} = -\sum_{i=1}^n i \cdot \underline{e}_i$ . By the same argument as in Subsect. 2.3,  $\gamma$  is nonnegative.

The adjoint representation of  $\mathfrak{k}$  on the linear span of  $\{X_{kl}^*\}$  is given by  $U(2\mathcal{E}_1)$ , hence we may choose for  $\underline{\lambda}$  the highest weight of any irreducible subrepresentation of the tensor product  $U(2\mathcal{E}_1) \otimes U(\underline{h})$ . By the Littlewood–Richardson rule, we may choose  $\underline{\lambda} = \underline{h} + \mathcal{E}_{r+1} + \mathcal{E}_{s+1}$ , where  $r$  and  $s \leq r$  are the heights of the first two columns of the Young diagram  $Y$  (i.e.,  $r$  is the smallest number such that  $h_{r+1} = h_\infty$  and  $s$  is the smallest number such that  $h_{s+1} \leq h_\infty + 1$ ). This choice of  $\underline{\lambda}$  gives the necessary condition for unitarity

$$r + s \leq 2h_\infty = N. \quad (3.9)$$

Next, we display a ground state  $|\underline{h}\rangle_F$  in the Fock space of  $N$  real scalar fields by

$$|\underline{h}\rangle_F = \left[ \prod_{k=1}^{m_1} a^{*\wedge r_k} \right]^0 |0\rangle, \quad (3.10)$$

where  $r_k$  are the heights of the columns of the Young diagram  $Y$  and  $a^{*\wedge r}$  stands for the component

$$a^{*\wedge r} = \det \left( a_i^{p*} \right)_{\substack{p=1,\dots,r \\ i=1,\dots,r}} \quad (3.11)$$

of the antisymmetric  $O(N)$  tensor  $\vec{a}_1^* \wedge \dots \wedge \vec{a}_r^*$  ( $r \leq N$ ), while  $[\dots]^0$  stands for the corresponding component of the traceless part of the product tensor.

The presence of this ground state implies that all representations with highest weights as specified above are realized in this Fock space, and are indeed unitary. The superselection sectors of the bilocal field  $V(x_1, x_2)$  are thus classified by the Cartan eigenvalues of  $\mathfrak{sp}(\infty, \mathbb{R})$ :

$$|\underline{h}\rangle = (m_1 + h_\infty, \dots, m_r + h_\infty, h_\infty, \dots) \quad \text{with } r + s \leq N, \quad (3.12)$$

where  $h_\infty = N/2$ ,  $m_1 \geq m_2 \geq \dots \geq m_r > 0$  are integers, and  $r$  and  $s$  are the heights of the first two columns of the Young diagram  $Y$  whose rows have lengths  $m_i$ .

The gauge group  $O(N)$  acts unitarily on the Fock space by leaving the vacuum invariant and transforming the creation operators  $\vec{a}^*$  like a vector. Therefore the ground state  $|\underline{h}\rangle_F$  belongs to the unitary representation of  $O(N)$  given by the Young diagram  $Y$ . In fact, it is a common highest-weight vector for the commuting actions of  $\mathfrak{sp}(\infty, \mathbb{R})$  and  $\mathfrak{so}(N)$  (the Lie algebra of the gauge group) on the Fock space.

By the unitarity bound (3.9), only those Young diagrams occur whose first two columns have total height  $r + s \leq N$ . It remains to convince oneself that such Young diagrams give precisely all irreducible unitary representations of  $O(N)$ . The standard labeling [4] of the unitary representations of  $O(N)$  is given by pairs  $(Y, \pm)$  where  $Y$  is a Young diagram with at most  $N/2$  rows determining the representation of the subgroup  $SO(N)$ , and  $\pm$  stand for the two representations of the quotient group  $O(N)/SO(N) \cong \mathbb{Z}_2$  given by the determinant. Note that  $(Y, +)$  is equivalent to  $(Y, -)$  iff  $N$  even and  $Y$  has exactly  $N/2$  rows.

Since the completely antisymmetric rank  $r$  tensor representation of  $O(N)$  whose diagram  $Y_r$  consists of a single column of height  $r$  is equivalent to  $\det \otimes Y_{N-r}$ , the representation with diagram  $Y$  (such that  $r + s \leq N$ ) is equivalent to  $(Y, +)$  if  $r \geq N/2$ , and to  $(Y', -)$  if  $r \leq N/2$ , where  $Y'$  arises from  $Y$  by replacing the first column of height  $r$  by a column of height  $N - r$ . One easily sees that this relabeling of the irreducible unitary representations is a bijection.

This proves the analog of Theorem 1.

Turning to the analog of Theorem 2, we proceed by exploiting the vanishing of the determinant operator (1.8) in every superselection sector. Since

$$X_{11} \cdots X_{N+1, N+1} D_{N+1}^* | \underline{h} \rangle = \left( \prod_{m=1}^{N+1} 2(2h_m - m + 1) \right) | \underline{h} \rangle, \quad (3.13)$$

the vanishing of  $D_{N+1}$  implies that

$$2h_m = m - 1 \quad \text{for some positive integer } m \leq N + 1. \quad (3.14)$$

In particular,  $N' := 2h_\infty$  is a nonnegative integer and  $N' \leq 2h_m = m - 1 \leq N$ .

As a consequence, the representation is realized on the Fock space of  $N'$  real scalar fields. Assuming that  $D_N$  does not vanish, we conclude that  $N \leq N'$ , hence  $N' = N$  and  $h_\infty = N/2$ . Convergence of the ground state energy requires  $g_i = h_\infty$  up to an irrelevant finite renormalization, thus proving the analog of Theorem 2.

## 4 Concluding remarks

Finding all representations of an algebra is often a highly nontrivial problem. Great progress has been made in the mathematical theory of highest-weight representations of Lie algebras, and these methods have been successfully exploited for the classification of unitary positive-energy representations (superselection sectors) of conformal QFT models in *two* space-time dimensions.

In *four* space-time dimensions these powerful methods were thought to be inapplicable, because scalar local quantum fields do not satisfy commutation relations of Lie type [2]<sup>3</sup>. However, *bilocal* quantum fields appearing in certain operator product expansions do have this property. By virtue of this observation, one can benefit from the theory of highest-weight modules of Lie algebras in order to study positive-energy representations in quantum field theory.

This article illustrates the approach on a class of nontrivial examples, thus building the connection between two important developments in physics and in mathematics that have taken place unaware of each other during the last decades.

One is the Doplicher–Haag–Roberts (DHR) theory of superselection sectors in the framework of algebraic QFT, which establishes the duality between sectors and gauge

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<sup>3</sup>There are examples of Poincaré covariant local Lie fields which violate, however, the spectrum condition; see [15].

symmetry (of the first kind). The other is the classification of highest-weight unitary modules of certain simple Lie algebras including  $\mathfrak{sp}(2n, \mathbb{R})$  and  $\mathfrak{su}(n, n)$ . These Lie algebras are found to be realized in the commutation relations of the simplest bilocal quantum fields occurring in globally conformal invariant QFT in  $D \geq 4$  (even) dimensions.

Obtaining by Lie algebra methods the explicit classification of unitary positive-energy representations of the commutation relations satisfied by the bilocal fields, we prove that they are all realized in a Fock space representation, corresponding to the Segal–Shale–Weil representation in mathematical terminology.

This outcome was expected from the corresponding abstract result obtained in the DHR theory. However, considerable technical difficulties are encountered in relating the field representations and their extensions with the representations of the corresponding nets. The merit of our study is that it gives an independent re-derivation of the DHR result directly in the field-theoretic framework for the special cases at hand. Moreover, we have shown that, with the canonical choice of the Hamiltonian, the embedded *Lie algebras*  $\mathfrak{u}(\infty, \infty)$  and  $\mathfrak{sp}(\infty, \mathbb{R})$  possess the same unitary positive-energy representations as the associative *field algebras*.

On the other hand, our result facilitates the program of classifying globally conformal invariant quantum field theories in four dimensions, because it indicates that without loss of generality one can “decouple” scalar free fields from a model [18].

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## A Mode expansions of local and bilocal fields

The conformal Hamiltonian  $H$  is a central element of the Lie algebra  $\mathfrak{so}(D) \oplus \mathbb{R}$  of the maximal compact Lie subgroup of the conformal group  $SO(D, 2)$ . The eigenfunctions of the conformal Hamiltonian form a basis of test functions over the *compactified* Minkowski space  $\overline{M} \cong (\mathbb{S}^{D-1} \times \mathbb{S}^1)/\mathbb{Z}_2$ , and each eigenspace is a finite-dimensional representation of  $SO(D)$ . In the complex parameterization [27, 16, 22] of  $\overline{M}$  given by<sup>4</sup>

$$\overline{M} \cong \{z \in \mathbb{C}^D : z = e^{i\tau} u, \tau \in \mathbb{R}, u \in \mathbb{S}^{D-1}\} \quad (\text{A.1})$$

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<sup>4</sup>The embedding of the Minkowski space  $M$  in  $\overline{M}$  reads  $z^j = \frac{2x^j}{1+x^2-2ix^0}$  ( $j = 1, \dots, D-1$ ) and  $z^D = \frac{1-x^2}{1+x^2-2ix^0}$ , where  $x = (x^0, x^1, \dots, x^{D-1})$  and  $x^2 = -(x^0)^2 + (x^1)^2 + \dots + (x^{D-1})^2$ . This is a conformal map belonging to the connected complex conformal group.

( $\Rightarrow z^2 = (z^1)^2 + \dots + (z^D)^2 = e^{2i\tau}$ ), these eigenfunctions are the Fourier polynomials

$$f_{n,\ell,\mu}(z) = (z^2)^n h_{\ell,\mu}(z) = e^{i(2n+\ell)\tau} h_{\ell,\mu}(u). \quad (\text{A.2})$$

Here  $n$  is an arbitrary integer and  $\{h_{\ell,\mu}(u)\}_{\mu=1}^{\mathfrak{h}_\ell}$  is a (real) basis of *spherical harmonics* on  $\mathbb{S}^{D-1}$ , i.e., homogeneous harmonic polynomials of degree  $\ell = 0, 1, \dots$ . The number  $\mathfrak{h}_\ell$  of spherical harmonics of degree  $\ell$  in  $D$ -dimensional spacetime equals

$$\mathfrak{h}_\ell = \frac{D-2+2\ell}{D-2+\ell} \binom{D-2+\ell}{D-2}. \quad (\text{A.3})$$

The  $z$ -parameterization of  $\overline{M}$  is conformally equivalent to the affine parametrization of the Minkowski space, and any GCI (poly)local field  $\phi(x)$  can be transformed to a conformally covariant field in the  $z$ -coordinates [16, 22]. We introduce a system of modes  $\phi_{n,\ell,\mu}$  of  $\phi(z)$  by

$$\phi_{n,\ell,\mu} := \phi[f_{n,\ell,\mu}] = \frac{1}{V} \int_{\overline{M}} \phi(z) f_{n,\ell,\mu}(z) d^D z, \quad (\text{A.4})$$

where  $d^D z$  is the (complex) volume form of  $\mathbb{C}^D$  restricted on the real submanifold  $\overline{M}$  (this is well-defined only in even space-time dimension  $D$  since otherwise  $\overline{M}$  is nonorientable), and  $V$  is the (pure imaginary) volume of  $\overline{M}$ .

One can write the collection of all modes of the field  $\phi(z)$  as a formal power series

$$\phi(z) = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{\mathfrak{h}_\ell} f_{n,\ell,\mu}(z) \phi_{-n-\ell-\frac{D}{2},\ell,\mu}. \quad (\text{A.5})$$

The complex integral over  $\overline{M}$  gives rise to a linear functional on the space of all formal power series of the above type, called the *residue* [1, Sect. 3] and given explicitly by

$$\text{Res}_z f_{n,\ell,\mu}(z) := \delta_{n,-\frac{D}{2}} \delta_{\ell,0}. \quad (\text{A.6})$$

We can write

$$\phi_{n,\ell,\mu} = \text{Res}_z \phi(z) f_{n,\ell,\mu}(z), \quad (\text{A.7})$$

provided we choose  $h_{\ell,\mu}(z)$  to be orthonormal with respect to the residue, i.e.,

$$\text{Res}_z (z^2)^{-\frac{D}{2}-\ell} h_{\ell,\mu}(z) h_{\ell',\mu'}(z) = \delta_{\ell,\ell'} \delta_{\mu,\mu'}. \quad (\text{A.8})$$

An important property of the residue is that it is translation invariant:

$$\text{Res}_z \partial_{z^\alpha} f(z) = 0, \quad \alpha = 1, \dots, D. \quad (\text{A.9})$$

In addition, it satisfies the Cauchy formula [1]

$$\text{Res}_z ((z-w)^2)^{-\frac{D}{2}}_+ f(z) = f(w) \quad \text{for } f(z) \in \mathbb{C}[[z]], \quad (\text{A.10})$$

where  $((z-w)^2)_+^n$  denotes the formal series resulting from the Taylor expansion of  $((z-w)^2)^n$  in  $w$  around 0.

It is possible to characterize GCI fields  $\phi(z)$  as formal power series of the above type, with properties equivalent to the Wightman axioms. The corresponding algebraic structure is a higher-dimensional *vertex algebra* [16, 22, 1].

For massless scalar fields of canonical scaling dimension  $d_0 = (D-2)/2$  in even space-time dimension  $D$ , only the modes with  $n = 0$  and  $n = -\ell - d_0$  contribute in (A.5), which correspond to solutions of the wave equation. The mode expansion for a pair of conjugate fields then can be conveniently written as (cf. [22]):

$$\begin{aligned}\varphi(z) &= \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{h_{\ell}} \left\{ (z^2)^{-\ell-d_0} \varphi_{\ell+d_0, \mu} + \varphi_{-\ell-d_0, \mu} \right\} h_{\ell, \mu}(z), \\ \varphi^*(z) &= \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{h_{\ell}} \left\{ (z^2)^{-\ell-d_0} \varphi_{\ell+d_0, \mu}^* + \varphi_{-\ell-d_0, \mu}^* \right\} h_{\ell, \mu}(z),\end{aligned}\tag{A.11}$$

where the modes  $\varphi_{\pm\ell, \mu}^{(*)}$  are conjugate to each other:

$$(\varphi_{\ell, \mu})^* = \varphi_{-\ell, \mu}^*.\tag{A.12}$$

This corresponds to the conjugation law  $(\varphi(\bar{z}))^* = (z^2)^{-d_0} \varphi^*(z/z^2)$  reflecting the fact that we work in a complex parameterization of the real compactified Minkowski space.

In terms of the modes  $\varphi_{\ell, \mu}^{(*)}$ , the canonical commutation relations

$$[\varphi(z), \varphi^*(w)] = ((z-w)^2)_+^{-d_0} - ((w-z)^2)_+^{-d_0}\tag{A.13}$$

become

$$[\varphi_{\ell+d_0, \mu}, \varphi_{-\ell'-d_0, \mu'}^*] = \frac{d_0}{\ell+d_0} \delta_{\ell, \ell'} \delta_{\mu, \mu'} = [\varphi_{\ell+d_0, \mu}^*, \varphi_{-\ell'-d_0, \mu'}],\tag{A.14}$$

and all other commutators vanish. This follows from (A.11) and the orthogonal harmonic decomposition of  $((z-w)^2)_+^{-d_0}$  [1, Sect. 3.3]:

$$((z-w)^2)_+^{-d_0} = \sum_{\ell=0}^{\infty} \sum_{\mu=1}^{h_{\ell}} \frac{d_0}{\ell+d_0} (z^2)^{-\ell-d_0} h_{\ell, \mu}(z) h_{\ell, \mu}(w).\tag{A.15}$$

Choosing an enumeration,  $n = n(\ell, \mu) \in \mathbb{N}$ , we define an infinite number of creation and annihilation operators  $a_n^{(*)}$  and  $b_n^{(*)}$  for states of positive and negative charge by setting

$$\begin{aligned}a_n &= \sqrt{\frac{\ell+d_0}{d_0}} \varphi_{\ell+d_0, \mu}, & b_n &= \sqrt{\frac{\ell+d_0}{d_0}} \varphi_{\ell+d_0, \mu}^*, \\ a_n^* &= \sqrt{\frac{\ell+d_0}{d_0}} \varphi_{-\ell-d_0, \mu}^*, & b_n^* &= \sqrt{\frac{\ell+d_0}{d_0}} \varphi_{-\ell-d_0, \mu}.\end{aligned}\tag{A.16}$$

These operators satisfy the canonical commutation relations

$$[a_m, a_n^*] = \delta_{m,n} = [b_m, b_n^*], \quad [a_m, b_n^{(*)}] = 0, \text{ etc.} \quad (\text{A.17})$$

The conformal Hamiltonian  $H$  and the charge operator  $Q$  are then expressed as

$$H = \sum_{n=1}^{\infty} \varepsilon_n (a_n^* a_n + b_n^* b_n), \quad Q = \sum_{n=1}^{\infty} (a_n^* a_n - b_n^* b_n) \quad (\text{A.18})$$

where  $\varepsilon_{n(\ell,\mu)} := \ell + d_0$  ( $\ell = 0, 1, \dots$ ) are the energy eigenvalues.

Introducing the notation  $h_{n(\ell,\mu)}(z) := \sqrt{\frac{d_0}{\ell+d_0}} h_{\ell,\mu}(z)$ , we rewrite (A.11) as follows:

$$\begin{aligned} \varphi(z) &= \sum_{n=1}^{\infty} h_n(z) \left\{ (z^2)^{-\varepsilon_n} a_n + b_n^* \right\}, \\ \varphi^*(z) &= \sum_{n=1}^{\infty} h_n(z) \left\{ a_n^* + (z^2)^{-\varepsilon_n} b_n \right\}. \end{aligned} \quad (\text{A.19})$$

Similarly, one can write the mode expansion of a complex bilocal field  $W(z_1, z_2)$  satisfying the commutation relations (1.3) as

$$\begin{aligned} W(z_1, z_2) &= \sum_{n_1, n_2=1}^{\infty} h_{n_1}(z_1) h_{n_2}(z_2) \left\{ X_{n_1 n_2} + (z_1^2)^{-\varepsilon_{n_1}} (z_2^2)^{-\varepsilon_{n_2}} X_{n_1 n_2}^* \right. \\ &\quad \left. + (z_1^2)^{-\varepsilon_{n_1}} \left( E_{n_2 n_1}^- - \frac{N}{2} \delta_{n_1, n_2} \right) + (z_2^2)^{-\varepsilon_{n_2}} \left( E_{n_1 n_2}^+ - \frac{N}{2} \delta_{n_1, n_2} \right) \right\}, \end{aligned} \quad (\text{A.20})$$

where  $X_{mn}^{(*)}$  and  $E_{mn}^{\pm} = (E_{nm}^{\pm})^*$  satisfy the commutation relations (2.3).

The mode expansion of a real bilocal field  $V(z_1, z_2)$  satisfying (1.4) looks exactly the same, but without the superscripts  $\pm$  on  $E_{nm}$  and with the symmetry  $X_{mn}^{(*)} = X_{nm}^{(*)}$ .

## B Stress-energy tensor and the conformal Lie algebra

The *stress-energy tensor* in any conformal field theory is expected to be a local tensor field that gives rise to the space-time symmetry generators when integrated against certain functions suggested by the (classical) Lagrangian field theory. These integrals are usually ill defined in general axiomatic QFT. But in the presence of GCI the theory can be extended to the compactified Minkowski space as we stated in the previous appendix. Then one can introduce rigorously the notion of a stress-energy tensor without any further assumptions.

We shall formulate the notion of a stress-energy tensor in a GCI QFT directly in the  $z$ -picture introduced in the previous appendix. It is a *symmetric tensor* field  $T_{\alpha\beta}(z) = T_{\beta\alpha}(z)$ , which is *traceless*:  $T_{\alpha\alpha}(z) = 0$ , and *conserved*:  $\partial_{z^\alpha} T_{\alpha\beta}(z) = 0$  (summation over repeated indices). It is assumed also to be a *quasiprimary* tensor field of a scaling

dimension equal to the space-time dimension  $D$ . These assumptions can be conveniently reformulated using the following generating function of  $T_{\alpha\beta}$ :

$$T(z; v) = T_{\alpha\beta}(z) v^\alpha v^\beta. \quad (\text{B.1})$$

Note that  $T(z; v)$  is a quadratic polynomial in  $v$  with coefficients that are operator-valued (formal) distributions.

Then, the above postulates for  $T_{\alpha\beta}$  read as follows:

$$\partial_v^2 T(z; v) = 0 \quad (\text{tracelessness}), \quad (\text{B.2})$$

$$\partial_z \cdot \partial_v T(z; v) = 0 \quad (\text{conservation law}). \quad (\text{B.3})$$

The statement that  $T_{\alpha\beta}$  is a quasiprimary tensor field reads:

$$[T_\alpha, T(z; v)] = \partial_{z^\alpha} T(z; v), \quad (\text{B.4})$$

$$[H, T(z; v)] = (z \cdot \partial_z + D) T(z; v), \quad (\text{B.5})$$

$$[\Omega_{\alpha\beta}, T(z; v)] = (z^\alpha \partial_{z^\beta} - z^\beta \partial_{z^\alpha} + v^\alpha \partial_{v^\beta} - v^\beta \partial_{v^\alpha}) T(z; v), \quad (\text{B.6})$$

$$[C_\alpha, T(z; v)] = (z^2 \partial_{z^\alpha} - 2 z^\alpha z \cdot \partial_z - 2D z^\alpha + 2 z \cdot v \partial_{v^\alpha} - 2 v^\alpha z \cdot \partial_v) T(z; v), \quad (\text{B.7})$$

where  $T_\alpha, H, \Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$  and  $C_\alpha$  are the generators of the conformal Lie algebra  $\mathfrak{so}(D, 2)$ , which satisfy the relations:

$$\begin{aligned} [H, \Omega_{\alpha\beta}] &= 0 = [T_\alpha, T_\beta] = [C_\alpha, C_\beta], \\ [\Omega_{\alpha_1\beta_1}, \Omega_{\alpha_2\beta_2}] &= \delta_{\alpha_1\alpha_2} \Omega_{\beta_1\beta_2} + \delta_{\beta_1\beta_2} \Omega_{\alpha_1\alpha_2} - \delta_{\alpha_1\beta_2} \Omega_{\beta_1\alpha_2} - \delta_{\beta_1\alpha_2} \Omega_{\alpha_1\beta_2}, \\ [H, T_\alpha] &= T_\alpha, \quad [H, C_\alpha] = -C_\alpha, \\ [\Omega_{\alpha\beta}, T_\gamma] &= \delta_{\alpha\gamma} T_\beta - \delta_{\beta\gamma} T_\alpha, \quad [\Omega_{\alpha\beta}, C_\gamma] = \delta_{\alpha\gamma} C_\beta - \delta_{\beta\gamma} C_\alpha, \\ [T_\alpha, C_\beta] &= 2 \delta_{\alpha\beta} H - 2 \Omega_{\alpha\beta}. \end{aligned} \quad (\text{B.8})$$

Since  $T_{\alpha\beta}$  is a tensor field it requires “tensor test functions”. For

$$f(z; v) = f_{\alpha\beta}(z) v^\alpha v^\beta, \quad f_{\alpha\beta}(z) \in \mathbb{C}[z, 1/z^2], \quad (\text{B.9})$$

we define

$$T[f] := \frac{1}{2} \text{Res}_z \left\{ f(z, \partial_v) T(z; v) \right\}. \quad (\text{B.10})$$

Using the residue technique of [1] (see Appendix A), one can derive the following statement.

**Proposition 1.** *Let  $T(z; v)$  be a local field defined by (B.1) and satisfying relations (B.2)–(B.4) in a vertex algebra, which is not assumed to be conformal in advance. Introduce the operators  $X := T[f_X]$  for  $X = T_{\alpha\beta}, H, \Omega_{\alpha\beta}, C_\alpha$  ( $\alpha, \beta = 1, \dots, D$ ), where*

$$\begin{aligned} f_X &= \frac{z \cdot v}{z^2} g_X(z, v), \quad g_{T_\alpha} = v^\alpha, \quad g_H = v \cdot z, \\ g_{\Omega_{\alpha\beta}} &= z^\alpha v^\beta - z^\beta v^\alpha, \quad g_{C_\alpha} = z^2 v^\alpha - 2 z^\alpha z \cdot v. \end{aligned} \quad (\text{B.11})$$



Then these operators obey the conformal Lie algebra relations (B.8) if and only if Eqs. (B.5)–(B.7) hold.

Given the bilocal field  $W(z, w)$  or  $V(z, w)$ , we can define the stress-energy tensor  $T(z; v)$  by applying to the bilocal field a second order differential operator  $\mathcal{D} = \mathcal{D}(\partial_z, \partial_w; v)$  and equating the arguments [18]:

$$T(z; v) = \mathcal{D} W(z, w)|_{w=z}, \quad \text{or} \quad T(z; v) = \frac{1}{2} \mathcal{D} V(z, w)|_{w=z}, \quad (\text{B.12})$$

where

$$(D - 1)\mathcal{D}(\partial_z, \partial_w; v) = d_0[(v \cdot \partial_z)^2 + (v \cdot \partial_w)^2] - D(v \cdot \partial_z)(v \cdot \partial_w) + v^2(\partial_z \cdot \partial_w) \quad (\text{B.13})$$

and  $d_0 := (D - 2)/2$ . Note that the second formula in (B.12) follows from the first for  $V(z, w) = W(z, w) + W(w, z)$ . It is an easy exercise to verify that the harmonicity of  $W(z, w)$  (or of  $V(z, w)$ ) implies both the tracelessness and the conservation of  $T(z; v)$ ; for instance,

$$\begin{aligned} (D - 1)(\partial_z + \partial_w) \cdot \partial_v \mathcal{D}(\partial_z, \partial_w; v) W(z, w) \\ = -2((v \cdot \partial_z) \partial_z^2 + (v \cdot \partial_w) \partial_w^2) W(z, w) = 0. \end{aligned} \quad (\text{B.14})$$

**Proposition 2.** *Let  $W(z_1, z_2)$  be given by (A.20). Then  $T(z; v)$  defined by (B.12) generates a representation of the conformal Lie algebra by (B.11). Furthermore,  $W(z_1, z_2)$  transforms under this representation as a scalar bilocal field of dimension  $(d_0, d_0)$ ; in particular,*

$$[H, W(z_1, z_2)] = (z_1 \cdot \partial_{z_1} + z_2 \cdot \partial_{z_2} + 2d_0) W(z_1, z_2). \quad (\text{B.15})$$

A similar statement is valid for  $V(z_1, z_2)$ .

The proposition shows that, if one wants to realize *both*  $\mathfrak{u}(\infty, \infty)$  and  $\mathfrak{so}(D, 2)$  in the state space of the theory, one cannot absorb the central term in (1.3) involving the constant  $N$  by a redefinition of the field  $W(x_1, x_2) \mapsto W(x_1, x_2) - (N/2)(\Delta_{1,2}^+ + \Delta_{2,1}^+)$ , without its reappearance in formula (B.12) for the generators of  $\mathfrak{so}(D, 2)$ .

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